PUTNAM PRACTICE SET 29

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Problem 1. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function with the property that whenever A, B, C and D are the vertices of a square in the cartesian plane, then $f(A) + f(B) + f(C) + f(D) = 0$. Find f.

Problem 2. Functions f, g and h are differentiable on the interval $I = (-1/10, 1/10)$ and on this interval, they satisfy the following relations:

$$f' = 2f^2gh + \frac{1}{gh} \text{ and } f(0) = 1$$

$$g' = fg^2h + \frac{4}{fh} \text{ and } g(0) = 1$$

$$h' = 3fgh^2 + \frac{1}{fg} \text{ and } h(0) = 1.$$

Find an explicit formula for $f(x)$ on the interval I.

Problem 3. Is there a finite abelian group G with the property that the product of the orders of its elements equals 2^{2021}?

Problem 4. Let S be a set of rational numbers such that
- $0 \in S$;
- if $x \in S$, then $1 + x \in S$ and also $x - 1 \in S$; and
- if $x \in S \setminus \{0, 1\}$, then $\frac{1}{x(x-1)} \in S$.

Must S contain all rational numbers?